# Cluster algebras and canonical base

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## Project

Relate a cluster algebra $\mathcal{A}$ to Lusztig's canonical base / perverse sheaves on the spaces of quiver representations, or related spaces

## Goal

$\mathcal{A}$ has a (dual) canonical base $\mathbb{B}$ containing all cluster monomials.
(In fact, in our example today, $\mathbb{B} = \{\text{cluster monomials}\}$)

Cor. • positivity of Laurent expansions with respect to any seed. ← cluster alg. side
• factorization of dual canonical base elements. ← canonical base side

## Why?

— Original motivation (Fomin–Zelevinsky)
— Canonical base elements should reflect various properties of quiver representations.
So, want to relate canonical base / tilting theory
cluster category
— Also gives a monoidal categorification (Hernandez–Leclerc), as the canonical base is the set of simple objects in an abelian category.

Today I restrict myself to the very special case:

- $\mathcal{A}$ = cluster algebra for $\left(\begin{array}{c} \text{frozen vertex} \\ \begin{array}{cccccc} 0 & 1 & 2 & & \ell-1 & \ell \\ \circ \to & \bullet \to & \bullet \to & \cdots & \bullet \to & \bullet \end{array} \end{array}\right)$ type $A_{\ell+1}$
- $y_0, y_1, \cdots, y_\ell$ : initial variables
- $y[\alpha]$ : cluster variable corresponding to a positive root $\alpha$ of $A_\ell$

In the article (0905.0002), I studied a cluster algebra associated with a bipartite quiver. As I want to avoid non-essential complications, I consider the above example, where the same technique applies.

Recently Kimura and Qin announce that they generalize my result to an acyclic cluster algebra.

Cluster category does not fit well with Lusztig's theory. So I enlarge the algebra, instead of the category

In this example, we use rather ad-hoc construction :

representations of the quiver $\underset{1}{\circ} \to \underset{2}{\circ} \to \underset{3}{\circ} \to \cdots \to \underset{\ell}{\circ} \to \underset{\ell+1}{\circ}$

Initial variable $y_i \longleftrightarrow$ indecomposable module $0 \cdots\cdots 0 \ \overset{i+1}{\mathbb{C}} \xleftarrow{1} \mathbb{C} \xleftarrow{1} \cdots \xleftarrow{1} \overset{\ell+1}{\mathbb{C}}$

Other cluster variables (and monomials) are given by Caldero–Chapton formula

Let $W = W(1) \oplus W(2) \oplus \cdots \oplus W(\ell)$ be a graded vector space over $\mathbb{C}$
Let $w_i := \dim W(i)$.

not $\ell+1$

Then
$$\boxed{y[W] = \frac{1}{y_1^{w_1} \cdots y_\ell^{w_\ell}} \sum_v \mathrm{Euler}(Gr_v(x)) \prod_i y_i^{v_{i-1} + w_{i+1} - v_{i+1}}}$$

where $Gr_v(x)$ = quiver Grassmann for a general representation $x$ such that the underlying vector space $= W$,
and $v \in \mathbb{Z}_{\geq 0}^{\ell}$ dimension of submodules.

$x$: indecomposable $\iff$ $y[W] = y[\alpha]$ is a cluster variable.

I use this CC formula to show that cluster monomials correspond to perverse sheaves on the space of quiver rep's.

## Graded quiver varieties (of type $A_1$)

Consider the opposite quiver $\underset{1}{\circ} \leftarrow \underset{2}{\circ} \leftarrow \cdots \leftarrow \underset{\ell}{\circ} \leftarrow \underset{\ell+1}{\circ}$

$W = W(1) \oplus W(2) \oplus \cdots \oplus W(\ell+1)$ : graded vector space over $\mathbb{C}$

$\mathbb{E}_W := \bigoplus_i \operatorname{Hom}(W(i+1), W(i)) \ni x = \bigoplus_{i=1}^{\ell} x_i \qquad W(1) \xleftarrow{x_1} W(2) \xleftarrow{x_2} W(3) \leftarrow \cdots \xleftarrow{x_\ell} W(\ell+1)$

$\circlearrowleft G_W = \prod GL(W(i))$ the space of quiver representations with bases

We introduce a closed subvariety: (this is an affine graded quiver variety of type $A_1$)

$$\mathfrak{M}_0^\bullet(W) := \{ x \in \mathbb{E}_W \mid x^2 = 0 \} \subset \mathbb{E}_W$$

We will study $G_W$-invariant (constructible) $\mathbb{Z}$-valued functions on $\mathfrak{M}_0^\bullet(W) \subset \mathbb{E}_W$

Let $K(\mathcal{Q}_W) =$ the set of all such functions.

<u>NB.</u> In the original article, I used constructible sheaves, instead of functions. This is necessary even here for the proof of our main result. But in this exposition, I suppose the audience is not familiar with sheaves, and use functions instead.
This has a drawback. I cannot explain what are perverse sheaves. I will only say they are nice constructible functions...

✲ $K(Q_W)$ has a basis $\{1_{\mathcal{O}(x)}\}$ consisting of characteristic functions of orbits $\mathcal{O}(x)$ through $x$.

Later we will move $W$. So we change the notation $1_{\mathcal{O}(x)}$ to $M_W(V)$, where $V$ is the graded vector space, defined by $V = \operatorname{Im} x$.

✲ $K(Q_W)$ has a nicer basis $\{IC_W(V)\}$, given by the simple perverse sheaf associated with $\mathcal{O}(x)$.

I don't explain what perverse sheaves are. We have

$$IC_W(V) = M_W(V) + \sum a_{V'} M_W(V') \quad \text{for some } a_{V'} \in \mathbb{Z}$$

with $M_W(V')$ corresponding to an orbit in the closure of $\mathcal{O}(x)$.

As notation suggests $K(Q_W)$ is the Grothendieck ring of an additive category $Q_W$. In fact, $\operatorname{Hom}(K(Q_W), \mathbb{Z})$ is the module category of a quasihereditary algebra $A_W$.
The dual of $\{IC_W(V)\}$ is a base given by simple modules.

The definition of $A_W$ is geometric, and in this particular case it is probably possible to give a presentation. But I don't know how to do in general.

☆ $\bigoplus_W K(Q_W)$ has a structure of cocommutative coalgebra :

Fix $W \twoheadrightarrow W'$ and set $W^2 := \mathrm{Ker}$. Consider the diagram

$$\begin{tikzcd} & \mathfrak{Z}_0(W';W^2) := \{ x \in \mathfrak{M}_0^\bullet(W) \mid x(W^2) \subset W^2 \} \arrow[dl, "\kappa"'] \arrow[r, hook, "\iota"] & \mathfrak{M}_0^\bullet(W) \\ \mathfrak{M}_0^\bullet(W') \times \mathfrak{M}_0^\bullet(W^2) & & \end{tikzcd}$$

Define $K(Q_W) \xrightarrow{\triangle} K(Q_{W'}) \otimes K(Q_{W^2})$ by $\boxed{\triangle \varphi := \kappa_! \, \iota^* \varphi}$

where $(\kappa_! \psi)(x) = \sum_{m \in \mathbb{Z}} m \cdot \mathrm{Euler}(\kappa^{-1}(x) \cap \psi(m))$

NB. $\bigoplus_W K(Q_W)$ is a cosubalgebra of Lusztig's construction of $U(n^-)$.

☆ We introduce an equivalence relation $\sim$ on the set of $\bigcup_W \{ IC_W(V) \mid V \}$ generated by $IC_W(V) \sim IC_{W^\perp}(0)$ where $W^\perp = \mathrm{Ker}\, x / \mathrm{Im}\, x$.

Then we define $\mathbf{R} = \{ (f_W) \in \prod_W \mathrm{Hom}(K(Q_W), \mathbb{Z}) \mid \langle f_W, IC_W(V) \rangle = \langle f_{W'}, IC_{W'}(V') \rangle \}$

if $IC_W(V) \sim IC_{W'}(V')$.

One can show $\mathbf{R}$ is compatible with the comultiplication $\Delta$.
Therefore $\mathbf{R}$ is an algebra.

It has a base dual to $\{[IC_W(0)] \mid W: \text{graded vector space}\}$.
Denote it by $\{L(W)\}$.

I mentioned that $K(Q_W)^* = K(\mathrm{mod}\,A_W)$. The idea for $\sim$ comes from the fact that there exists a Hopf algebra $\mathcal{U}$ ($U_q(\widehat{\mathfrak{sl}_2})$: quantum affine $\mathfrak{sl}_2$ in our example) and a family of homomorphisms

$$\begin{tikzcd} \mathcal{U} \arrow[r, two heads] \arrow[dr, two heads] & A_W \arrow[d] \\ & A_{W'} \end{tikzcd}$$

compatible with $\Delta$ :

$$\begin{tikzcd} \mathcal{U} \arrow[r] \arrow[d, "\Delta"'] \arrow[dr, phantom, "\circlearrowright"] & A_W \arrow[d, "\Delta"] \\ \mathcal{U} \otimes \mathcal{U} \arrow[r] & A_{W^1} \otimes A_{W^2} \end{tikzcd}$$

We have $\mathbf{R} \simeq K(\mathrm{mod}\,\mathcal{U})$, and $L(W)$ is the class of a simple module.

Thus we have a monoidal categorification of $\mathbf{R}$.

<u>Goal</u> $\mathbf{R} \cong \mathcal{A}$ so that $L(W) \longleftrightarrow$ a cluster monomial corresponding to a generic representation of $E_W$

In order to relate **R** with the cluster algebra, we introduce several other spaces:

$\mathbb{E}_W^* :=$ dual space to $\mathbb{E}_W = \bigoplus_i \mathrm{Hom}(W(i), W(i+1))$ $\qquad W(1) \xrightarrow{x_1^*} W(2) \xrightarrow{x_2^*} \cdots \xrightarrow{x_\ell^*} W(\ell+1)$

Choose another graded vector space $V = V(1) \oplus \cdots \oplus V(\ell+1)$.

$$\mathrm{Gr}_V(W) := \{ (x^*, \underset{\substack{\uparrow \\ \text{I-graded subsp}}}{S} \subset W) \mid S \cong V, \ x^*(S) \subset S \}$$

$$\begin{array}{c} \mathrm{Gr}_V(W) \\ \downarrow \pi^\perp \\ \mathbb{E}_W^* \end{array} \qquad x^* = \begin{array}{cc} S & W/S \\ \end{array} \begin{bmatrix} * & * \\ 0 & * \end{bmatrix}$$

$\boxed{\text{fiber of } \pi^\perp = \text{quiver Grassmann}}$

$\mathrm{Gr}_V(W)$ is a vector bundle over the product $\prod_i \mathrm{Gr}(v_i, W_i)$ of (usual) Grassmann manifolds. $(v_i = \dim V(i))$

It is a subbundle of a trivial bundle $\mathbb{E}_W^* \times \prod_i \mathrm{Gr}(v_i, W_i)$.

We consider annihilator:

$$\mathfrak{M}^\bullet(V, W) := \{ (x, S \subset W) \in \mathbb{E}_W \times \prod_i \mathrm{Gr}(v_i, W_i) \mid \langle x, x^* \rangle = 0 \ \ \forall x^* \text{ s.t. } x^*(S) \subset S \}$$

$$= \{ (x, S \subset W) \mid \mathrm{Im}\, x \subset S \subset \mathrm{Ker}\, x \} \qquad x = \begin{bmatrix} 0 & * \\ 0 & 0 \end{bmatrix}$$

Let $\pi : \mathfrak{M}^\bullet(V,W) \longrightarrow \mathbb{E}_W$ : natural projection

Note $x^2 = 0$ if $x \in$ image of $\pi$.

Thus $\pi : \mathfrak{M}^\bullet(V,W) \longrightarrow \mathfrak{M}^\bullet_0(W)$.

These are graded quiver varieties of type $A_1$.

nonsingular / affine

<u>NB</u>, $W(i-2) \xleftarrow{x_i} W(i) \xleftarrow{x_{i+1}} W(i+1)$

$\operatorname{Im} x_{i+1} \subset V(i) \subset \operatorname{Ker} x_i$

$\therefore$ $\boxed{\text{fiber of } \pi = \prod_i Gr\left(v_i - \operatorname{rk} x_{i+1}, \operatorname{Ker} x_i / \operatorname{Im} x_{i+1}\right)}$

Let $\pi_W(V) := \pi_!(1_{\mathfrak{M}^\bullet(V,W)}) \in K(Q_W)$.

<u>Lemma</u> $\mathrm{ch} : R \longrightarrow \mathbb{Z}^{\star}$  $\star$ = all graded vector spaces $V = \mathbb{Z}^{\ell+1}_{\geq 0}$

$L(W) \longmapsto \{\langle \pi_W(V), L(W) \rangle\}$ is injective.

Therefore it is enough to calculate

$$\langle \pi_W(V), L(W) \rangle = \text{coeff. of } IC_W(0) \text{ in } \pi_!(1_{m^\bullet(V,W)})$$

$$= m_0 \quad \text{where} \quad \pi_!(1_{m^\bullet(V,W)}) = \sum_{V'} m_{V'} IC_W(V')$$

## Key Observation

$\pi_W(V)$ is related to CC formula via Fourier transform $\Phi : \mathrm{Func}(E_V) \longrightarrow \mathrm{Func}(E_V^*)$.

$$\boxed{\Phi(\pi_W(V)) = \pi^\perp_!(1_{\mathrm{Gr}_V(W)})}$$

$$\begin{tikzcd}
\mathrm{Gr}_V(W) \arrow[d, "\pi^\perp"'] \arrow[r, leftrightarrow, bend left, "\text{annihilator}"] & m^\bullet(V,W) \arrow[d, "\pi"] \\
E_W^* \arrow[r, leftrightarrow, "\text{dual}"'] & E_W
\end{tikzcd}$$

Recall $\pi^\perp_!(1_{\mathrm{Gr}_V(W)})(x^*) = \mathrm{Euler}(\mathrm{Gr}_V(x^*))$.

$\mathrm{Gr}_V(x^*)$: quiver Grassmann

If $x^*$ is general, RHS appears in CC formula.

The Fourier transform $\Phi$ is defined by $\Phi(\varphi) = p_{2!}(p_1^*\varphi \cdot 1_P)$

where

$$\begin{tikzcd}
& E_V \times E_V^* \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \supset P = \{(x, x^*) \mid \langle x, x^* \rangle \leq 0\} \\
E_V & & E_V^*
\end{tikzcd}$$

It is known that $\Phi$ maps a simple perverse sheaf to a simple perv. sheaf.

$\therefore$ $\langle \pi_W(V), L(W) \rangle =$ coeff. of $\pi_!^{\perp}(1_{gr_V(W)})$ in $\Phi(IC_W(0))$

Now $IC_W(0) = 1_{\{0\}}$ (analog of $\delta$-function)

$\therefore$ Fourier transform $\Phi(IC_W(0)) = 1_{E_W^*}$ (constant function).

All other $\Phi(IC_W(V')) =$ char. func. of an orbit $+ \sum$ smaller
have smaller support $\neq E_W^*$

$\Longrightarrow$ $\langle \pi_W(V), L(W) \rangle = \pi_!^{\perp}(1_{gr_V(W)})(x^*)$

general element

$\underset{\text{CC formula}}{\Longrightarrow}$ $L(W)$ is a cluster monomial.